HALF-SPACE GREEN'S FUNCTION FOR LAMB'S PROBLEM

Maryam ERFANINIA
Ph.D. Student, IIEES, Tehran, Iran
maryam.erfaninia@iiees.ac.ir

Mohsen KAMALIAN
Professor, IIEES, Tehran, Iran
kamalian@iiees.ac.ir

Keywords: 3D Green's function, Source image, Half space, Heaviside step function, Dirac delta function

The wave propagation in an isotropic, homogeneous linear elastic half-space is treated. The half-space is a semi-infinite region that bounded by a horizontal plane. The vibration is generated by an arbitrary direction buried point pulse. It is necessary to consider both of source and receiver point are located in the interior of the three dimensional domain. The solution of this elastodynamic problem, i.e. the so-called Lamb's problem, is derived by using the method of source image as well as the superposition principle. Accordingly the transient response of the problem in time domain can be considered as the superposition of the responses to real and imaginary sources in the full-space and some additional vertical loads on the surface of the half-space. The additional vertical loads distributed over a rectangular area on the surface of a half-space are space and time-dependent functions that vary with time as Heaviside step, Dirac delta and derivatives of Dirac delta functions. The motion at depth produced by a point source applied on the surface when the time variation of the pulse is $H(t)$, $\delta(t)$ and $\dot{\delta}(t)$ is obtained based on the approach used by Eringen et al., 1977. To achieve Laplace transform displacement, they have employed Helmholtz potentials for displacement field and satisfy Laplace transform wave equation as well as Hankel transform of boundary condition. The time-domain solutions are found via a modified version of Cagniard-Pekeris method by Emami and Eskandari-Ghadi (2019).

The solutions obtained in this way satisfy the traction-free boundary conditions over the surface of the half-space. These Green's functions can be implemented in 3D time-domain BEM and no discretization of the ground surface is needed. The efficiency of 3D site response analysis of topographic features by the BEM is improved and the computational time and cost is reduced.

Statement of the problem and its solution

An isotropic, homogeneous linear elastic half-space with an arbitrary direction buried point pulse in a Cartesian coordinate system is considered as a problem. The governing equation can be expressed as:

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla\nabla.\mathbf{u}+\rho\mathbf{b}=\rho\,\partial^2\mathbf{u}\big/\partial^2 t \quad , \quad \mathbf{b}(\mathbf{x},t)=\delta(t)\delta(\mathbf{x}-\boldsymbol{\xi})\mathbf{e} \quad , \quad \mathbf{t}(r,0,t)=0 \tag{1}$$

in which $\hat{\boldsymbol{\imath}}$ and $\mathbf{x}$ are the location vector of source and receiver, respectively. The sum of the three displacement field satisfying Equation 1 can be written as Equation 2:

$$(\,)^{g}=(\,)^{rf}_{\boldsymbol{\xi}}+(\,)^{if}+(\,)^{al} \tag{2}$$

where $(\,)^{rf}$, $(\,)^{if}$ and $(\,)^{al}$ stand for fundamental solution of real source, fundamental solution of imaginary source and half-space response to additional load, respectively. Full space displacement field due to impulsive point load and image of that with respect to the central surface ($x_3=0$), $(\,)^{mf}$ and $(\,)^{if}$, can be written as Equation 3:

$$U_{ij}(\mathbf{x},t;\boldsymbol{\xi})=\frac{t}{4\pi\rho r^2}\left\{\begin{aligned}&\left(\frac{3r_ir_j}{r^3}-\frac{\delta_{ij}}{r}\right)\left[H\left(t-\frac{r}{c_1}\right)-H\left(t-\frac{r}{c_2}\right)\right]+\\&\frac{r_ir_j}{r^2}\left[\frac{1}{c_1}\delta\left(t-\frac{r}{c_1}\right)-\frac{1}{c_2}\delta\left(t-\frac{r}{c_2}\right)\right]+\frac{\delta_{ij}}{c_2}\delta\left(t-\frac{r}{c_2}\right)\end{aligned}\right\}\rightarrow\left\{\begin{aligned}&for\ (\,)^{rf}\ ,\ \mathbf{x}=(x_1,x_2,x_3)\\&for\ (\,)^{if}\ ,\ \mathbf{x}=(x_1,x_2,-x_3)\\&r_i=x_i-\xi_i\end{aligned}\right. \tag{3}$$

The half-space solution for additional vertical load, $(\,)^{al}$ can be obtained by using superposition principle as expressed by Equation 4:

$$U_{ij}\left(\boldsymbol{x},t;\bar{\boldsymbol{x}}\right)=\sum_{k=1}^{8}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_i^k\left(\bar{\boldsymbol{x}}\right)W_j^k\left(\boldsymbol{x},t;\bar{\boldsymbol{x}}\right)\,d\bar{x}_1 d\bar{x}_2 \qquad (4)$$

where $W_j^k\left(\boldsymbol{x},t;\bar{\boldsymbol{x}}\right)$ is the half-space displacement field due to a point load applied on the surface. These point loads are included vertical unit load varying with time as $H(t), tH(t), \delta(t)$ and $\dot{\delta}(t)$ functions.

Additional vertical loads due to surface tractions from real and imaginary sources will be as Equation 5:

$$\begin{aligned} t_{i3} &= f_i^1 H(t-\frac{\bar{r}}{c_2}) + f_i^2 H(t-\frac{\bar{r}}{c_1}) + f_i^3 (t-\frac{\bar{r}}{c_2}) H(t-\frac{\bar{r}}{c_2}) + f_i^4 (t-\frac{\bar{r}}{c_1}) H(t-\frac{\bar{r}}{c_1}) \\ &+ f_i^5 \delta(t-\frac{\bar{r}}{c_2}) + f_i^6 \delta(t-\frac{\bar{r}}{c_1}) + f_i^7 \dot{\delta}(t-\frac{\bar{r}}{c_2}) + f_i^8 \dot{\delta}(t-\frac{\bar{r}}{c_1}) \quad , \quad \bar{r}_i = \bar{x}_i - \xi_i \end{aligned} \qquad (5)$$

where $\bar{\boldsymbol{x}}$ is the location vector of surface. $f_i^1, f_i^2, \ldots, f_i^8$ are functions varying with velocity wave (c_1), pulse direction, $\hat{\boldsymbol{\imath}}$ and $\bar{\boldsymbol{x}}$. W_j^k in transform domain is derived by using a technique developed by Eringen et al. (1977). To take the inverse transforms and obtain time domain response, the modified version of Cagniard-Pekeris method by Emami and Eskandari-Ghadi (2019) is applied. The time domain displacement field in vertical direction due to $H(t-\frac{\bar{r}}{c_1})$ is presented by Equation 6 for instance:

$$u_3 = \left\{ N_1(\boldsymbol{x}, t-\frac{\bar{r}}{c_1}) + N_2(\boldsymbol{x}, t-\frac{\bar{r}}{c_1}) \right\} H(t-\frac{\bar{r}}{c_1}) \ , \ y^2 = \underbrace{\left[(x_1-\bar{x}_1)^2 + (x_2-\bar{x}_2)^2 \right]}_{\tilde{r}^2} \xi^2 + \frac{x_3^2}{c_2^2}$$

$$N_1\left(\boldsymbol{x},t\right) = \frac{2}{\pi}\int_1^{\infty} \frac{I\left[\Psi_1 \Psi_1' g_1(\Psi_1)\right]}{\sqrt{\xi^2-1}} d\xi \quad , \quad g_1(\Psi_1) = \frac{-F}{2\pi\mu}\frac{1}{c_2}\frac{\alpha_1(2\Psi_1^2+\frac{1}{c_2^2})}{\left[(2\Psi_1^2+\frac{1}{c_2^2})^2 - \frac{4}{c_2^2}\alpha_1\alpha_2\Psi_1^2\right]}$$

$$N_2\left(\boldsymbol{x},t\right) = \frac{2}{\pi}\int_1^{\infty} \frac{I\left[\Psi_2 \Psi_2' g_2(\Psi_2)\right]}{\sqrt{\xi^2-1}} d\xi \quad , \quad g_2(\Psi_2) = \frac{F}{\pi\mu}\frac{1}{c_2}\frac{\alpha_1\Psi_2^2}{\left[(2\Psi_2^2+\frac{1}{c_2^2})^2 - \frac{4}{c_2^2}\alpha_1\alpha_2\Psi_2^2\right]}$$

$$\Psi_m(t) = \begin{cases} 0 & \frac{\bar{r}}{c_1} < t < \frac{x_3}{c_1} + \frac{\bar{r}}{c_1} \\ \frac{i}{y^2}(\xi\tilde{r}(t-\frac{\bar{r}}{c_1}) - \frac{x_3}{c_2}\sqrt{\alpha^2 y^2 - (t-\frac{\bar{r}}{c_1})^2}) & \frac{x_3}{c_1} + \frac{\bar{r}}{c_1} < t < \alpha y + \frac{\bar{r}}{c_1} \\ \frac{1}{y^2}(i\xi\tilde{r}(t-\frac{\bar{r}}{c_1}) + \frac{x_3}{c_2}\sqrt{(t-\frac{\bar{r}}{c_1})^2 - \alpha^2 y^2}) & \alpha y + \frac{\bar{r}}{c_1} < t < \infty \end{cases} \quad , \quad \begin{aligned} m=1 &\rightarrow \alpha = \frac{c_2}{c_1} \\ m=2 &\rightarrow \alpha = 1 \end{aligned} \qquad (6)$$

REFERENCES

Emami, M. and Eskandari-Ghadi, M. (2019). Transient Interior Analytical Solutions of Lamb's Problem. *To be Appeared in Mathematics and Mechanics of Solids*.

Eringen, A.C., Suhubi, E.S., and Bland, D.R. (1977). Elastodynamics, vols. 1 and 2. *Physics Today, 30*, 65.